Dynamic Programming

Dynamic Programming

- **D*ynamic Programming* is a general algorithm design technique**
- **for solving problems defined by or formulated as recurrences with overlapping subinstances**

- **Invented by American mathematician Richard Bellman in the 1950s to solve optimization problems and later assimilated by CS**

- **"Programming" here means "planning"**

- **Main idea:**
  - **set up a recurrence relating a solution to a larger instance to solutions of some smaller instances**
  - **- solve smaller instances once**
  - **record solutions in a table**
  - **extract solution to the initial instance from that table**

Example: Fibonacci numbers (cont.)

Computing the n^{th} Fibonacci number using bottom-up iteration and recording results:

$F(0) = 0$
$F(1) = 1$
$F(2) = 1+0 = 1$
…
$F(n-2) =$
$F(n-1) =$
$F(n) = F(n-1) + F(n-2)$

0	1	1	. . .	$F(n-2)$	$F(n-1)$	$F(n)$

Efficiency:
- time n
- space n

What if we solve it recursively?

Examples of DP algorithms

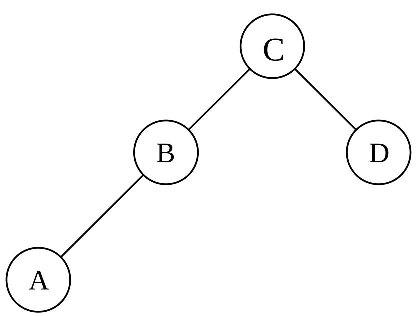
- **Computing a binomial coefficient**
- **Longest common subsequence**
- **Warshall's algorithm for transitive closure**
- **Floyd's algorithm for all-pairs shortest paths**
- **Constructing an optimal binary search tree**
- **Some instances of difficult discrete optimization problems:**
  - **traveling salesman**
  - **knapsack**

Optimal Binary Search Trees

Problem: Given n keys $a_1 < \ldots < a_n$ and probabilities $p_1, \ldots, p_n$ searching for them, find a BST with a minimum average number of comparisons in successful search.

Since total number of BSTs with n nodes is given by $C(2n,n)/(n+1)$, which grows exponentially, brute force is hopeless.

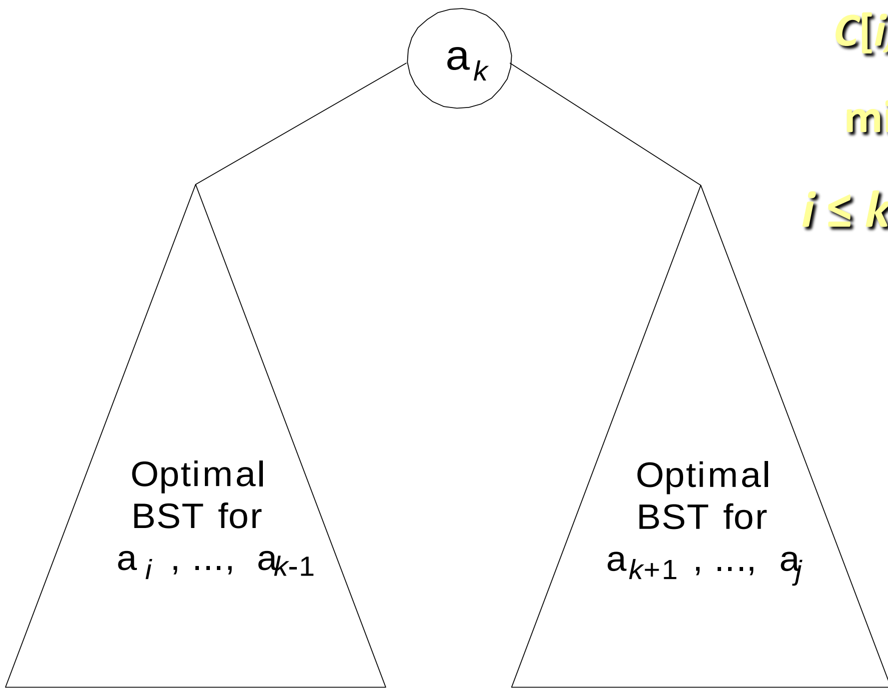
Example: What is an optimal BST for keys *A*, *B*, *C*, and *D* with search probabilities 0.1, 0.2, 0.4, and 0.3, respectively?



Average # of comparisons = 1*0.4 + 2*(0.2+0.3) + 3*0.1 = 1.7

DP for Optimal BST Problem

Let $C[i,j]$ be minimum average number of comparisons made in T[i,j], optimal BST for keys $a_i < \ldots < a_j$, where $1 \le i \le j \le n$. Consider optimal BST among all BSTs with some a_k ($i \le k \le j$) as their root; T[i,j] is the best among them.



$$C[i,j] = \min_{i \le k \le j} \{p_k \cdot 1 + \sum_{s=i}^{k-1} p_s (\text{level } a_s \text{ in } T[i,k-1] + 1) + \sum_{s=k+1}^{j} p_s (\text{level } a_s \text{ in } T[k+1,j] + 1)\}$$

DP for Optimal BST Problem (cont.)

After simplifications, we obtain the recurrence for $C[i,j]$:

$$C[i,j] = \min_{i \le k \le j} \{C[i,k-1] + C[k+1,j]\} + \sum_{s=i}^{j} p_s \quad \text{for } 1 \le i \le j \le n$$

$$C[i,i] = p_i \quad \text{for } 1 \le i \le j \le n$$

Example: key A B C D

probability 0.1 0.2 0.4 0.3

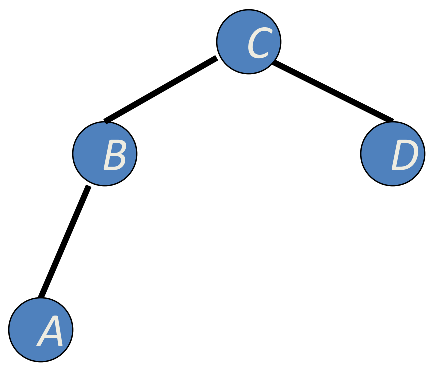
The tables below are filled diagonal by diagonal: the left one is filled using the recurrence

$$C[i,j] = \min_{i \le k \le j} \{C[i,k-1] + C[k+1,j]\} + \sum_{s=i}^{j} p_s, \quad C[i,i] = p_i;$$

the right one, for trees' roots, records k's values giving the minima

i \ j	0	1	2	3	4
1	0	.1	.4	1.1	1.7
2		0	.2	.8	1.4
3			0	.4	1.0
4				0	.3
5					0

i \ j	0	1	2	3	4
1		1	2	3	3
2			2	3	3
3				3	3
4					4
5					



optimal BST

Optimal Binary Search Trees

ALGORITHM *OptimalBST*($P[1..n]$)

```
//Finds an optimal binary search tree by dynamic programming
//Input: An array P[1..n] of search probabilities for a sorted list of n keys
//Output: Average number of comparisons in successful searches in the
//          optimal BST and table R of subtrees' roots in the optimal BST
for i ← 1 to n do
    C[i, i − 1] ← 0
    C[i, i] ← P[i]
    R[i, i] ← i
C[n + 1, n] ← 0
for d ← 1 to n − 1 do //diagonal count
    for i ← 1 to n − d do
        j ← i + d
        minval ← ∞
        for k ← i to j do
            if C[i, k − 1] + C[k + 1, j] < minval
                minval ← C[i, k − 1] + C[k + 1, j]; kmin ← k
        R[i, j] ← kmin
        sum ← P[i]; for s ← i + 1 to j do sum ← sum + P[s]
        C[i, j] ← minval + sum
return C[1, n], R
```

Analysis DP for Optimal BST Problem

Time efficiency: $\Theta(n^3)$ **but can be reduced to** $\Theta(n^2)$ **by taking advantage of monotonicity of entries in the root table, i.e.,** $R[i,j]$ **is always in the range between** $R[i,j-1]$ **and** $R[i+1,j]$

Space efficiency: $\Theta(n^2)$

Method can be expanded to include unsuccessful searches

Application of dynamic programming

- Longest common subsequence problem
- Checker board
- Bio-informatics
- Matrix chain multiplication

Scope of research

- Linear search problem

Assignment

- Q.1)Differentiate Dynamic Programming with Divide and conquer method.
- Q.2)Compare Dynamic Programming with Greedy Method.
- Q.3) State the advantages of OBST over BST with example.